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The influence of shear on mixing across density interfaces

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This paper discusses critically observations of mixing processes across density interfaces in laboratory experiments and inferences that have been made and can be made from these observations. Fluxes of heat or salt and entrainment velocities have been found to depend on negative powers of an overall Richardson number Ri^* based on the buoyancy jump across the interface, the depth of the homogeneous layer and the intensity of the turbulence near the source. When the Reynolds and Péclet numbers are large, the fluxes or entrainment velocities appear to be proportional to the minus one and minus three-halves powers of Ri^* for flows with and without mean shear respectively, and this difference has caused speculation about the accuracy of the experimental data and about the reasons for the two laws if the difference is real. In the present discussion, we accept the accuracy of the two laws and attribute the higher entrainment rate for shear flows to the decrease of r.m.s. velocities near the interface with increasing Ri^* in the case of zero shear. A plausible argument yields the unifying result that the entrainment rates in both cases are proportional to Ri^{-1}, where Ri is a Richardson number based on the buoyancy jump and velocities and lengths characteristic of the turbulence near the interface. It is suggested that the $Ri^{-\frac{3}{2}}$ behaviour inferred by Turner is based on an erroneous interpretation of experimental data.

In the course of the discussion, it is shown that the drag coefficient in flow of a stratified fluid over a rough surface is independent of the Richardson number (or density jump across the interface or inversion) and depends only on the ratio of the roughness length to the depth of the homogeneous layer. This has obvious implications for problems of parameterizing the momentum flux near the ground in the atmosphere.

1. Introduction

Geophysicists and engineers have a considerable interest in problems concerning the erosion and motion of interfaces between fluids of different densities and the related problem of heat or salt flux across these surfaces. Such interfaces occur frequently in the atmosphere and in oceans, lakes and reservoirs. An example of the practical importance of these studies is artificial destratification of reservoirs to improve water quality in the hypolimnion (stagnant region below

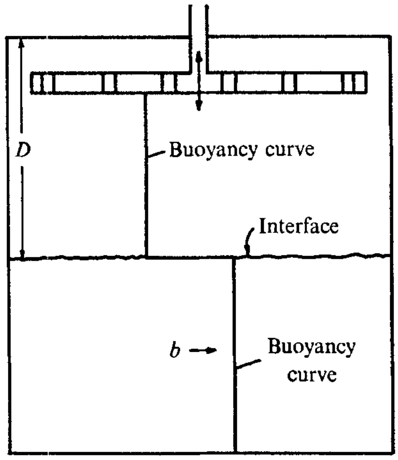

FIGURE 1. Mixing experiment of Rouse & Dodu (1955) and Turner (1968). In the figures of this paper the interface is shown as a discontinuity in density. Actually it is a thin layer with a thickness of 1 cm or so, independent of the Richardson number.

the thermocline). One method involves the pumping of fluid from the hypolimnion through a tube and discharging the water in a jet downward from the free surface. The heavy discharged water moves down to the interface and the mixing in the upper layer and the turbulence generated by the jet erode the interface, causing it to weaken, move downward and eventually disappear (Brush 1970).

Geophysical implications of mixing across density interfaces are numerous. In the oceans, for example, suddenly increased stress forces exerted by the wind at the water–air surface will cause the upper mixed layer to increase in depth at a rate dependent on the stress, the instantaneous depth of the layer, the density jump across the interface and, perhaps, other effects (Kato & Phillips 1969). In the atmosphere, inversions are common and the motion of these surfaces and heat, momentum and moisture fluxes across them are of great importance to our understanding of atmospheric turbulence and its parameterization in numerical models.

2. Experiments without shear

A number of laboratory experiments have been designed to help give an understanding of the problem. The first was by Rouse & Dodu (1955) and involved a vessel with two layers of liquid of different densities, as in figure 1. A grid of solid bars was oscillated vertically with a small stroke a in the upper layer and observations were made of the *entrainment velocity* u_e, or the downward velocity of propagation of the interface. This experiment is characteristic of those without shear. Shear, of course, is common in natural circumstances and is a source of turbulent kinetic energy.

Cromwell (1960) performed a similar experiment to simulate the pycnocline, but the first reliable data were obtained by Turner (1968). Turner ran two different experiments. One was as described above except that the stirring was in the lower layer; the other had stirring in both layers. In the first experiment, fluid was withdrawn from the stirred layer at a rate adjusted to keep the interface at the same distance from the grid, and the gradual fall of the air–water interface was ignored. The entrainment velocity was then defined by $Au_e = Q$, where Q is the volume withdrawn per unit time and A is the cross-sectional area of the tank. In the second experiment, both layers were turbulent and, with the same stirring action by the two grids, the interface stayed at the mid-level.

Theoretical considerations of these experiments involve the concept of buoyancy, defined as $b = (\rho - \rho_0)\, g/\rho_0$, where ρ is the density and ρ_0 is a reference value. In all experiments, the density difference was small and this justifies the use of the Boussinesq approximation (Boussinesq 1903; Spiegel & Veronis 1960). This means that b is the only quantity involving density or gravity entering the analysis.

In the one-grid experiment with the upper level mixed, if the interface is allowed to move downward at speed u_e (no fluid added or subtracted), the buoyancy flux q satisfies the equation

$$\partial q/\partial z = \partial \bar{b}/\partial t, \tag{1}$$

where $\bar{b}$ is the mean buoyancy in the upper layer and z may be measured downwards from the top of the upper layer. Since $\bar{b}$ varies very little with height† in the upper layer, the flux at the interface is

$$q = -D d(\Delta b)/dt, \tag{2}$$

where Δb is the buoyancy jump across the interface and we have put $q = 0$ at $z = 0$. Thus

$$q = -d(D\Delta b)/dt + u_e \Delta b. \tag{3}$$

Mass conservation shows that $D\Delta b$ is constant, so that $q = u_e \Delta b$. This may be used to define an entrainment velocity when both layers are agitated. In this case, if ρ_0 is the average of the two densities, the buoyancy flux at the interface is

$$q = -\tfrac{1}{2} D\, d(\Delta b)/dt$$

and we may define the entrainment velocity to be

$$u_e = -\frac{D}{2\Delta b}\frac{d}{dt}(\Delta b). \tag{4}$$

There have been a number of recent experiments of a similar type, for example by Brush (1970). Equipment identical to that of Turner was constructed by Wolanski (1972), and the one- and two-grid experiments were run with stratification caused by heat, salt, sugar and suspensions of sediments and minute silica spheres. Additional experiments have been run in Turner's apparatus by Linden (1973), Crapper (1973) and Crapper & Linden (1974) using heat and salt.

† The variation of $\bar{b}$ with height results in a relative error of order $\overline{\Delta b}/\Delta b$, where $\overline{\Delta b}$ is the increment in buoyancy across the 'homogeneous' layer. This is observed to be very small whenever density interfaces exist. We give its order of magnitude in equation (37) for experiments with and without shear.

An important result of the experiments by Turner (1968) may be expressed as

$$u_e/\omega = C[\omega^2/(\Delta b)]^n, \tag{5}$$

where ω is the frequency of the oscillating grid and C is independent of ω and Δb. A number of lengths were kept constant in the experiment, so that the dimensional quantity C may be a function of these. Turner found that, for larger values of $\Delta b/\omega^2$, the exponent $n = 1$ when stratification is related to temperature differences and $n = \frac{3}{2}$ when it is related to differences in salt content. Later investigations have confirmed these results and, very recently, Crapper & Linden (1974) have shown rather convincingly that the difference in the values of n is due to the influence of the relatively large molecular conductivity in the heating experiments (the coefficient K is much smaller for salt). C. G. H. Rooth (Turner 1973) in unpublished work has found a $\frac{3}{2}$ dependence in heating experiments when larger turbulent velocities are generated, so that it appears well established that the $\frac{3}{2}$ dependence is appropriate for larger Péclet numbers $Pe = u_1 l_1/K$, where u_1 and l_1 are velocity and length units. Crapper & Linden suggest a threshold value of $Pe \simeq 200$ when u_1 and l_1 are characteristic of the turbulence near the interface. The dependence on the Reynolds number Re has not been established because of the rather small ranges of Re in the experiments, but both Wolanski (1972), who varied Re by a factor of 3, and Crapper & Linden (1974) report very weak dependence, if any. We may, therefore, write for large Pe and Re and strong stability

$$u_e/\omega = C_1\omega^3/(\Delta b)^{\frac{3}{2}}, \tag{6}$$

where C_1 is a function of a, D and a_1, a_2, ..., where a is the total stroke of the oscillating grid and $a_1, a_2, \ldots$, are lengths characteristic of the grid and its location. It is convenient to introduce a dimensionless quantity K_1 by the definition

$$C_1 = \frac{a^4}{D^{\frac{3}{2}}} K_1\left(\frac{a}{D}, \frac{a}{a_1}, \frac{a}{a_2}, \ldots\right), \tag{7}$$

and therefore
$$u_e/u_* = K_1 Ri^{*-\frac{3}{2}}, \quad Ri^* = D\Delta b/u_*^2, \tag{8}$$

where $u_* = \omega a$. It is likely (see § 7) that K_1 is independent of a/D when this ratio is small.

3. Experiments with shear

Several experiments have been constructed to introduce shearing currents into turbulent density-stratified systems in an effort to simulate atmospheric and oceanic phenomena. The first of these of direct relevance to our discussion was that of Kato & Phillips (1969). The apparatus was a large circular annular channel filled with salt water with an initially linear density gradient. A constant stress $\tau = u_*^2$ was applied by rotating a flat screen at the surface (figure 2). They found

$$u_e/u_* = K_2 Ri^{*-1}, \tag{9}$$

where Ri^* is of the same form as in (8) and Δb is the buoyancy jump from the upper mixed layer to the quiescent region below.

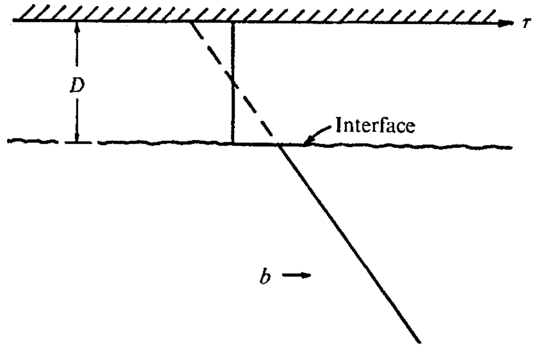

FIGURE 2. Idealized experiment of Kato & Phillips (1969). The initial buoyancy profile is linear. ——, profile at time t when the mixed layer has a depth D.

It is important for later purposes to present an analysis of the drag coefficient $2u_*^2/U^2$, where U is the speed of the screen in the experiment of Kato & Phillips. They found that U/u_* increased with time (or depth) with u_* held fixed. At first glance one might expect this to be an influence of the stable density distribution in the fluid system, but on closer consideration it seems more reasonable to neglect Δb entirely and consider the flow and turbulence in the upper layer as turbulent flow due to the motion of a rough plate at $z = 0$ with the interface at $z = D$ serving only to reduce the mean velocity to zero at that level. This is supported by a description by Kato & Phillips: "The movement [of a line of hydrogen bubbles] indicated that the mean velocity varied most rapidly near the screen and near the entrainment interface, being almost constant in the central region, where the velocity was typically about half that of the screen." The mean motion seems to have been very close to that in turbulent plane Couette flow (Robertson 1959) and a theory for the ratio U/u_* may be obtained by use of the technique of Izakson (1937) and Millikan (1938). We assume for the mean velocity near the screen (in a co-ordinate system moving with the screen)

$$\bar{u}/u_* = f(z/z_0), \tag{10}$$

where z_0 is the roughness length, and, in the interior, the velocity-defect law (Monin & Yaglom 1971, p. 298)

$$(\tfrac{1}{2}U - \bar{u})/u_* = g(z/D).$$

Writing $U/2u_* = m(z_0/D)$ and matching the two solutions near $z = z_0$ in the usual way, we find

$$\frac{U}{u_*} = -\frac{2}{\kappa}\ln\frac{z_0}{D} + A, \tag{11}$$

where κ is von Kármán's constant and A is another constant. Kato & Phillips give data on $U(t)$ for some of their experiments but, although the paper contains an empirical equation for $D(t)$, it does not agree with the data over the whole time period of the experiment. However, one case permits a comparison and this is shown in figure 3, in which the theoretical curve is

$$U/u_* = (2/\kappa)\ln D(t) + 5{\cdot}78, \tag{12}$$

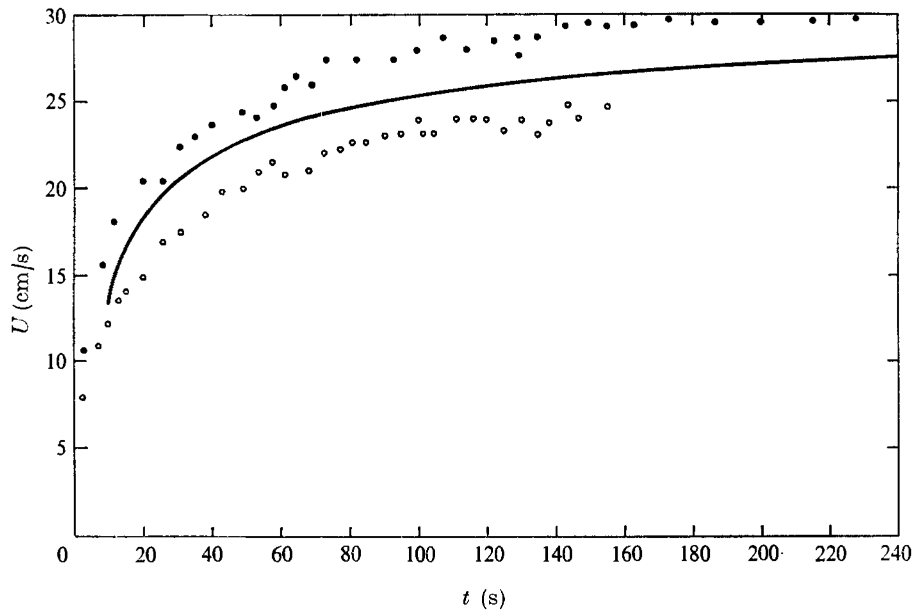

FIGURE 3. The solid curve is based on the theory of (11), in which $D(t)$ is given by curve II, figure 5 of Kato & Phillips (1969), with $\tau = 2{\cdot}12$. The data points are from figure 3 of Kato & Phillips: ●, $\tau = 2{\cdot}75$; ○, $\tau = 1{\cdot}49$. All experiments have the same buoyancy gradient. The stresses are in $\mathrm{cm^2/s^2}$.

where D is expressed in centimetres. The single constant was chosen to give $U = 27{\cdot}25$ cm/s at $t = 200$ s. The agreement is remarkable and leaves little doubt† that U/u_* is independent of the Richardson number, which varied by a factor of 100 in the course of the experiments. In subsequent discussions, we suppress any dependence of quantities on z_0/D.

The argument may be reasonably extend ed to flow of air over the cool surface of the earth. An inversion will be present at some height and the momentum stress will be a function only of the depth, the roughness length and the wind speed at the height of the inversion. This has obvious usefulness in problems of parameterizing the momentum flux near the ground in numerical atmospheric models.

An experiment by Moore & Long (1971) was constructed to permit a steady state. In a large channel shaped like a race track, fluid was injected from nearly horizontal jets at the bottom (salt water) and top (fresh water) in opposite directions to obtain a shearing current (figure 4). Zero mean vertical velocities were achieved by withdrawing equal volumes of fluid through numerous holes in the bottom and top. The system permitted measurement of the buoyancy flux q in a steady-state situation. At larger values of the density difference, two homogeneous layers existed at the top and bottom with an interface in the middle. The experiment yielded

$$q = K_3(\Delta u)^3/D, \tag{13}$$

† It also leaves little doubt that a recent suggestion to the contrary by the author is wrong (Long 1973).

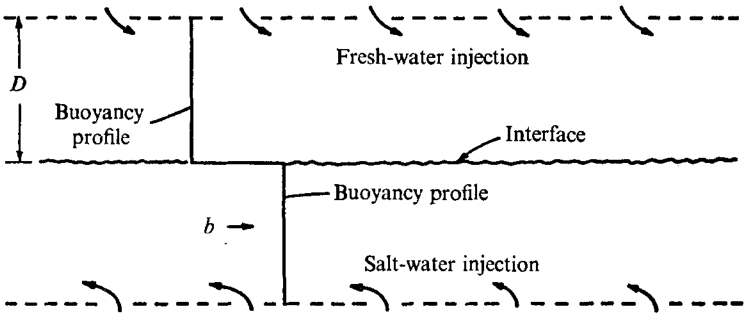

FIGURE 4. Idealized experiment of Moore & Long (1971). In order to balance the inflow through the jets, an equal volume of fluid was extracted from numerous holes on the top and bottom.

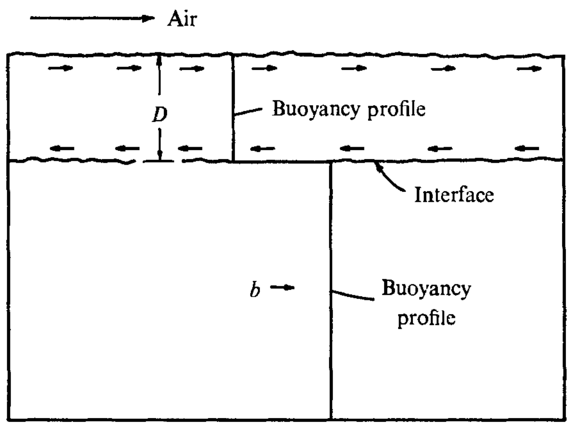

FIGURE 5. Mixing experiment of Wu (1973). The air blowing over the free surface induces the turbulence and the mean velocities in the upper layer as indicated by the small arrows.

where Δu is the difference between the mean velocities measured near the top and bottom. If we define the entrainment velocity by $u_e \Delta b = q$, equation (13) yields the same result as that in Kato & Phillips [equation (9)] if, as seems very likely from the discussion of the Kato & Phillips experiment, $\Delta u/u_*$ is independent of the Richardson number, where u_*^2 is the constant momentum flux in the tank. Moore & Long also ran unsteady experiments similar to those of Kato & Phillips with a fluid with an initially linear buoyancy gradient and subject to the system of jets and withdrawals at the bottom only. They obtained a result equivalent to (9).

Finally, in a recent experiment by Wu (1973), the source of energy and shear was a current of air blowing over a vessel containing a two-fluid system. The apparatus and the shear produced are shown in figure 5. Wu also obtained (9) although his coefficient of proportionality was much smaller.

4. Comparison of experiments with and without shear

The different dependence on Ri^* for the two experiments has been the source of perplexity (Turner 1973; Linden 1973) because the mixing processes appear to be very similar. Indeed, Linden (1973) has stated that the Kato & Phillips data are also consistent with a $-\frac{3}{2}$ behaviour, although support for this statement seems lacking. In the rest of this paper, we attempt to contribute to a unified understanding of the two results.

Turner (1973) has made the valuable suggestion that the erosion of the interface should depend on the properties of the turbulence near the interface, in particular on the r.m.s. velocity u_1 and the integral length scale l_1 near the interface. Thus he proposed the form

$$u_e/u_1 = f(Ri), \quad Ri = l_1 \Delta b/u_1^2, \tag{14}$$

where possible dependence on other quantities is suppressed and it is assumed that Pe and Re are large. In an attempt to determine the dependence on Ri from his density-interface experiments, *in which u_1 and l_1 were not measured,* Turner used unpublished experimental data by Thompson (see Turner 1973; Crapper & Linden 1974 for a description). Thompson used Turner's apparatus *with a homogeneous fluid* and one grid. He measured u and l at many levels, where u is the r.m.s. velocity and l is the integral length scale at a depth z. As reported by Crapper & Linden, Thompson found that l increased linearly with distance from the grid but was independent of ω. He also found that u decreased with z but at a given z was proportional to ω. Although Thompson's experiment had no density variation, Turner (1973), Thorpe (1973) and Crapper & Linden (1974) have assumed that his results are directly applicable to the mixing experiments. Thus, at $z = D$, they used

$$\frac{u_1}{\omega a} = C_2\left(\frac{a}{D}, \frac{a}{a_1}, \frac{a}{a_2}, \ldots\right), \quad \frac{l_1}{D} = C_3\left(\frac{a}{a_1}, \frac{a}{a_2}, \ldots\right), \tag{15),(16}$$

so that (8) could be written as

$$\frac{u_e}{u_1} = K_4 Ri^{-\frac{3}{2}}, \quad K_4 = K_4\left(\frac{a}{D}, \frac{a}{a_1}, \frac{a}{a_2}, \ldots\right). \tag{17}$$

Turner thus obtained the same exponent for the dependences on Ri^*, defined in (8), and on Ri, defined in (14). Notice that the proportionality of u_1 and ω follows from dimensional analysis *but only when the fluid is homogeneous.*

We may also obtain a dependence on Ri for the shearing experiments. With shear, we have

$$\partial\tau/\partial z = \partial\bar{u}/\partial t, \tag{18}$$

where $\bar{u}$ is the mean horizontal velocity at a depth z. In the steady-state experiments of Moore & Long (1971), $\partial\tau/\partial z = 0$ and therefore τ is constant with height. Since $\tau = -\overline{u'w'}$, and since the correlation coefficient is very likely to be of order one in the homogeneous layers, it follows that $u_* = \tau^{\frac{1}{2}}$ is proportional to u_1. If we use $l_1 \sim D$, we obtain

$$u_e/u_1 = K_5 Ri^{-1} \tag{19}$$

for the Moore & Long experiment. In the experiment of Kato & Phillips, we may use (18) to obtain the increment in τ over the depth D:

$$\Delta\tau/\tau \sim UD/Tu_*^2, \tag{20}$$

where T is the time period for a change of depth of order D, so that $T \sim D/u_e$. Therefore,

$$\frac{\Delta\tau}{\tau} \sim \frac{U}{u_*}\frac{u_e}{u_*} \sim Ri^{*-1}. \tag{21}$$

This shows that the stress varies very little over the depth, so that $u_* \sim u_1$, and (19) again holds.

Thus two different entrainment velocities are indicated in the two cases even when the characteristics of the eroding eddies are the same, and this is more perplexing than the difference in the exponent of Ri^*. The subsequent discussion of this paper questions the applicability of Thompson's experiment, in particular (15), to an experiment with a density interface and suggests that (17) is incorrect.† The difficulty is indicated by a simple analysis based on the assumption that (19) is correct with or without shear. If we accept (8) for the case without shear and use $l_1 \sim D$, we obtain

$$u_1/\omega a = K_6 Ri^{*-\frac{1}{6}} \tag{22}$$

instead of $u_1 \propto \omega a$ as was inferred from Thompson's experiments. Thus, if the assumptions are correct, there is a very weak dependence on the Richardson number which could not, of course, have been revealed by Thompson's experiments. Such a weak dependence is, nevertheless, capable of accounting for the difference in the power laws.

5. Energy arguments

The dependence of u_1/u_* on Ri^* in experiments without shear, as indicated in (22), may also be obtained by a plausible argument based on energy considerations. When there is shear, experiment indicates that

$$q \sim u_1^3/D \sim u_*^3/D. \tag{23}$$

Let us now evaluate q in the homogeneous layer near the interface. We get $q \sim u_1 b_1$, where b_1 is the r.m.s. buoyancy fluctuation, and we make the plausible assumption that the correlation is of order one. Thus

$$u_1^2/b_1 D \sim 1, \tag{24}$$

so that the kinetic energy $T' \sim u_1^2$ and available potential energy $V' = \frac{1}{2}b_1 l \sim b_1 D$ (Long 1970) are of the same order.‡ This is perhaps surprising but, if true in the shearing experiments, it should also be true when shear is absent. Assuming

† Linden (1973) has attempted to derive the $Ri^{-\frac{3}{2}}$ law by order-of-magnitude arguments. If we accept the conclusions of the present paper, Linden's argument must also be incorrect.

‡ We mean that the two energies are proportional. The constant of proportionality is very small, as we see in § 6.

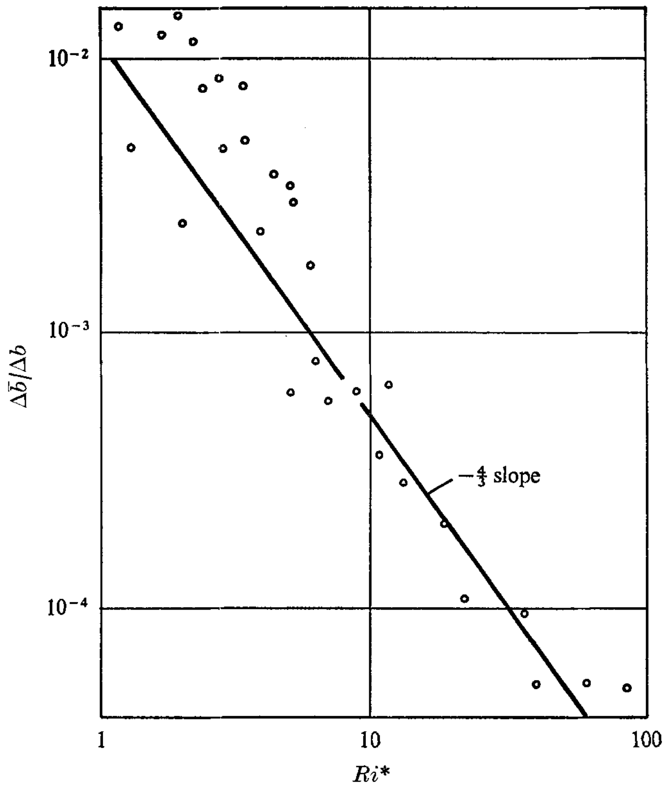

FIGURE 6. Non-dimensional ratio of the buoyancy difference across a 'homogeneous' layer and the buoyancy jump across the interface. The data are from Wolanski (1972).

this, since $q \sim u_1 b_1 \sim u_e \Delta b$ in this case, we have

$$u_e/u_* \sim u_1^3/Du_*\Delta b \sim u_*^3/(D\Delta b)^{\frac{3}{2}}, \tag{25}$$

or

$$u_1/u_* \sim Ri^{*-\frac{1}{6}} \tag{26}$$

as before. This 'derivation' is not independent of the arguments leading to (22) because we have again assumed the behaviour in (8) and (9), and the Ri^{*-1} law at least is controversial (Linden 1973). It is possible, however, to offer independent evidence in favour of the relation (24), or $T' \sim V'$, and thus to improve the argument. As we discuss in §6, other experimental and theoretical findings indicate that the flux Richardson number $q/\tau\bar{u}_z$ tends to a constant when there is shear and strong stability, and this behaviour implies (24). Equation (26) then follows from the less controversial $Ri^{*-\frac{3}{2}}$ law in experiments without shear.

The decrease in r.m.s. velocity with an increase in Richardson number when density variations are present as in (26) may be caused by the weak density gradient in the layers that we have called 'homogeneous'. In a layer as a whole, the slight density variation still has dynamic importance, as is indicated by the proportionality of kinetic energy and available potential energy. Such arguments have been advanced earlier by the author (Long 1972, 1973).

An additional relationship may be obtained for the experiments without shear. If we assume that the small buoyancy difference $\overline{\Delta b}$ across the 'homogeneous' layer is of the order of the r.m.s. buoyancy fluctuation (implying an eddy length scale of order D), (24) and (26) lead to

$$\overline{\Delta b}/\Delta b \sim Ri^{*-\frac{1}{3}}. \tag{27}$$

This quantity was measured by Wolanski (1972) for his salt experiments (figure 6). There is good agreement with (27), especially at higher values of Ri^*, although the scatter is probably too great to distinguish between the Ri^{-1} law and the $Ri^{-\frac{3}{2}}$ law.

The energy argument may be amplified. Rouse & Dodu (1955) and others (Kato & Phillips 1969; Turner 1973; Wu 1973) have suggested that the Ri^{*-1} law implies that the change in potential energy is proportional to the energy supply by the external source. On this basis, the $Ri^{*-\frac{3}{2}}$ law would not conform to any simple energy considerations. The above discussion indicates that the last conclusion is not correctly drawn. The energy equation for experiments with or without shear may be written as

$$\frac{\partial}{\partial t}\overline{(\tfrac{1}{2}c'^2)} = -\frac{\partial}{\partial z}\overline{[w'(\tfrac{1}{2}c'^2+p'/\rho_0)]}+\tau\bar{u}_z+q-\epsilon, \tag{28}$$

where p' is the turbulent pressure, c' is the turbulent speed, ϵ is the dissipation function and $\bar{u}_z = 0$ in experiments without shear. In the shearing experiments, the velocity difference is proportional to $\tau^{\frac{1}{2}}$, and the two energy-source terms, as well as the dissipation, are of order u_1^3/D or u_1^3/l_1 near the interface. If $q \sim u_e\Delta b$ is also of this order, we obtain $\bar{u}_e/u_1 \sim Ri^{-1}$ as in (19). When shear is absent, the single source term is the first term on the right-hand side of (28) and is also of order u_1^3/l_1. The Ri^{-1} law again implies equality of all sink and source terms. The correct interpretation of experimental results thus seems to be that the turbulence has a character that causes potential energy to increase at a rate proportional to the rate at which kinetic energy is supplied *to the region of the interface* and not necessarily proportional to the rate of generation of kinetic energy at the external source.

6. Implications of the energy arguments

We have reached the conclusion that in all experiments in which there is strong stability, so that turbulence in part of the region is intermittent, all terms in the energy equation are proportional to each other. When there is shear we may express this, in part, by saying that the flux Richardson number Ri_f assumes a constant (critical) value Ri_{fc}. This may also be characteristic of geophysical fluid systems; for example, Kullenberg (1971) has assumed this behaviour to obtain a theoretical expression for the eddy coefficient of heat diffusion that agrees very well with data in the sea. Ri_{fc} may be computed from the experimental data of Kato & Phillips (1969). If we use the mean buoyancy flux $\bar{q} = \frac{1}{2}u_e\Delta b$ in the mixed layer, the nearly constant momentum flux τ in the mixed layer, the velocity difference across the layer and the empirical result $u_e/u_* = 2{\cdot}5Ri^{*-1}$, we obtain

$$Ri_{fc} = \frac{\bar{q}}{\tau\bar{u}_z} = \frac{Du_e\Delta b}{2u_*^2 U} = 1{\cdot}2\frac{u_*}{U}. \tag{29}$$

For the higher velocities U/u_* varies very slowly and we adopt the value 20 for this ratio as indicated by the data of Kato & Phillips. This yields $Ri_{fc} \simeq 0{\cdot}06$, which is close to the value 0·05 adopted by Kullenberg on the basis of his observations in the sea. These low values of Ri_{fc} may be contrasted with estimates three times as big by Ellison and others (Turner 1973), and Kullenberg finds that data of Ellison & Turner (1960) and Bowden (1960) are not consistent with the lower values.

When the overall Richardson number is of order one or less, experiments indicate that the fluid is fully turbulent (Arya & Plate 1969; Moore & Long 1971). The observations of Arya & Plate show that the flux Richardson number increases as the Richardson number increases but with a tendency to level off for $Ri > 0{\cdot}1$ at a value $Ri_f = 0{\cdot}06$. This is in excellent agreement with the values cited above, but this may be because U/u_* happens to be similar in the experiments of Kato & Phillips and Arya & Plate and in the measurements of Kullenberg.

There is a difficulty with the concept of a critical flux Richardson number above which turbulence is supposed to die out. In fact, as we have seen, turbulence exists at infinite values of Ri_f in experiments without shear. The reason, of course, is that there is another source of kinetic energy, namely the energy flux divergence term of (28). The apparent success of the flux Richardson number concept probably arises from the tendency for all energy terms to be proportional to u^3/l, and this suggests the use of the quantity

$$Ri_f^* = q/(u^3/l) \tag{30}$$

instead of Ri_f. Since $q \sim ub$, we find that Ri_f^* is of the order of the ratio of available potential energy to turbulent kinetic energy. We have already argued that this has an upper limit whether shear is present or not, so that an upper limit Ri_{fc}^* exists in both cases and indeed, from physical considerations, may well have the same value in experiments with or without shear and in natural circumstances in the atmosphere and oceans. We may estimate Ri_{fc}^* as

$$Ri_{fc}^* \simeq 0{\cdot}35V'/T',$$

where V' is the available potential energy $\frac{1}{2}bl$ and T' is the kinetic energy. We have taken the correlation coefficient between w' and b' as 0·3, $w \simeq 0{\cdot}6u$ and $v \simeq 0{\cdot}75u$. In the case of shear we estimate Ri_{fc} from the experiment of Kato & Phillips:

$$Ri_{fc} = \frac{\bar{q}}{\tau\bar{u}_z} \simeq 63\frac{V'}{T'}\frac{u_*}{U} \simeq 1{\cdot}2\frac{u_*}{U},$$

where we have taken $D \simeq 14l$, in accordance with measurements in a pipe (Schlichting 1955), $u_*^2 \simeq 0{\cdot}3uw$ and $T' \simeq u^2$. Thus $V'/T' \simeq 0{\cdot}020$ and $Ri_{fc}^* \simeq \frac{1}{150}$. If this critical value of Ri_f^* is universal, it would be a more useful concept than Ri_{fc}, which, according to (29), varies with the drag coefficient.

The concept of the Monin–Oboukhov length $L = \tau^{\frac{3}{2}}/q\kappa$ (Monin & Yaglom 1971, p. 427) may be re-examined in the light of our discussion of experiments with density interfaces. In the mixed layer $\tau \sim u^2$ and $q \sim ub$, so that

$$L \sim u^2/b. \tag{31}$$

Since $u^2 \sim bl$ in the layer, $l \sim L$, that is, L is proportional to the eddy size. We also found $l \sim D$, so that the mixed layer has a depth proportional to the Monin–Oboukhov length. We may find the constant of proportionality from the experiment of Kato & Phillips (1969). If we use the average flux of buoyancy in the mixed layer $\bar{q}$, we have $Ri_{fc} = qD/\tau U$. It follows from (29) that

$$L/D = \tau^{\frac{3}{2}}/\bar{q}\kappa D = (\tfrac{1}{2}c_d)^{\frac{1}{2}}/\kappa Ri_{fc} \simeq 2, \tag{32}$$

where c_d is the drag coefficient $2u_*^2/U^2$. Alternatively we may compute L from the definition using the estimates for Ri_{fc}^*. We get $L/l \simeq 30$ and again $L/D \simeq 2$. Kitaigorodskii (1960) found $L/D \simeq 1{\cdot}2$ for the mixed layer in the ocean and recently Sundaram (1973) has computed $L/D \simeq 4$ for a lake.

Finally we remark that in both of the basic experiments considered in this paper $q \sim u_e \Delta b \sim u_1 \overline{\Delta b}$, so that

$$\frac{u_e}{u_1} \sim \frac{\overline{\Delta b}}{\Delta b} \sim \frac{l_1 \overline{\Delta b}/u_1^2}{l_1 \Delta b/u_1^2} \sim Ri^{-1}\frac{V'}{T'}. \tag{33}$$

The existence of turbulence implies that $V' \leqslant T'$, so that the experimental result $u_e/u_1 \sim Ri^{-1}$ in both experiments shows that u_e/u_1 is a maximum consistent with the maintenance of the turbulence.

7. Discussion of an idealized experiment

It is instructive to discuss an idealized experiment, which has limiting behaviours close to the two basic experiments. This is a two-fluid system (figure 7) with a plate or screen oscillating back and forth in its plane. The densities are nearly constant in each layer and the upper layer presumably deepens with time. The oscillation is produced by applying to the plate a stress $\tau = u_*^2$ which is constant in magnitude over each half-cycle. The amplitude of the oscillation is a and may be large or small. If the Péclet and Reynolds numbers are large, it seems reasonable to assume that

$$u_e/u_* = f(Ri^*, a/D), \tag{34}$$

where $D\Delta b$ in Ri^* is constant from considerations of mass continuity. If we use the classical case of flow over a flat plate as a guide (Monin & Yaglom 1971, p. 311), it seems likely that f is independent of viscosity in the case of a smooth plate and of the nature of the roughness in the case of a rough plate.

If we let $a/D \to \infty$, u_e/u_* will become independent of a/D and we shall obtain an experiment similar to that of Kato & Phillips (1969). Their results and the similar experiments by Moore & Long (1971) and by Wu (1973) indicate that

$$u_e/u_* = K_7 Ri^{*-1}. \tag{35}$$

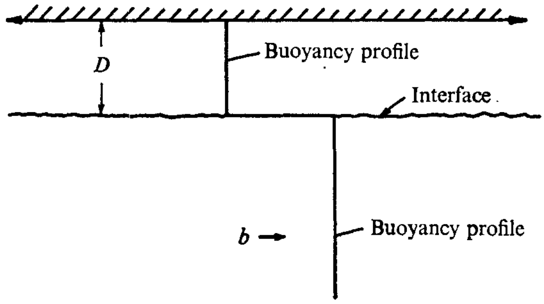

FIGURE 7. Idealized mixing experiment with turbulence produced by a plate oscillating in its own plane.

On the other hand, if we let $a/D \to 0$, we should again find independence of a/D and, since this is similar to Turner's experiment, we may write

$$u_e/u_* = K_8 Ri^{*-\frac{3}{2}}. \tag{36}$$

We are again led to two different laws for the entrainment velocity, but this simple experiment indicates that this should not be considered paradoxical since the two limits $a/D \to 0$ and $a/D \to \infty$ are very different. Notice that, in a given experiment with fixed a of moderate size, the erosion is at first rapid as in (35), slowing down gradually and tending to the lower rate in (36) as D increases.

A variation of this idealized experiment is useful to apply the ideas of § 5 to gain an appreciation of the inevitability of the appearance of homogeneous layers. In this case, there is fluid above and below the oscillating plate, which now is porous to permit a buoyancy flux through it. The initial distribution has uniform buoyancies $-\frac{1}{2}\Delta b$ above and $+\frac{1}{2}\Delta b$ below the plate. At the beginning of the experiment, eddies of small dimensions $h \sim u_* t$ form near the plate. The Richardson number $Ri = h\Delta b/u_1^2$ is small, so that buoyancy is unimportant dynamically and the erosion proceeds rapidly. Ri is increasing with time, however, and, when Ri is of order one, $V' \sim b_1 h \sim T' \sim u_1^2$. Since $b_1 \sim \overline{\Delta b}$, we have $\overline{\Delta b}/\Delta b \sim Ri^{-1} \sim 1$, so that the density difference across the mixing layer is not yet small. As time goes on Ri eventually becomes large. However, $V' \sim T'$, so that with an increase in depth b_1 or $\overline{\Delta b}$ must decrease and the homogeneous layer forms. The behaviour, as we have seen, is

$$\overline{\Delta b}/\Delta b \sim Ri^{*-1} \quad \text{or} \quad Ri^{*-\frac{4}{3}} \tag{37}$$

depending on the relative amplitude of the oscillation.

8. Summary

This discussion attempts to give a coherent interpretation of the experimental measurements of entrainment rates u_e across density interfaces. If we define a Richardson number Ri^* based on the density jump across the interface, the friction velocity or stirring rate imposed externally and the depth of the homogeneous layer, the experiments indicate different variations of u_e with Ri^* depending on the presence or absence of shear. We find this to be reasonable because the erosion should depend on the characteristics of the turbulence near the interface and the intensity of the turbulence near the interface decreases with Ri^* when shear is absent. Finally, it is shown that the available potential energy and the kinetic energy of the turbulence are of the same order in the shearing experiments in the homogeneous layer. With the mild assumption that this is also true in the experiments without shear, it follows that entrainment velocities are proportional to Ri^{-1} in both cases, where Ri is the Richardson number expressed in terms of the buoyancy jump and velocities and lengths characteristic of the turbulence near the interface. This discussion indicates a need for observations of r.m.s. velocities near the interface to compare with the prediction of this paper that u_1 is proportional to $\omega^{\frac{4}{3}}$ rather than to ω as measured by Thompson.

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